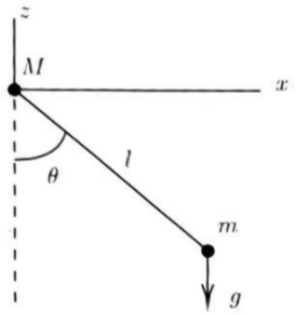
第十七届丘成桐大学生数学竞赛（数学物理赛道）
个人解答

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1 Problem 1

Consider a sliding pendulum, consisting of a mass M which can move without friction along a horizontal bar, and which is connected by a massless rod of length l to another mass m.



1. Derive the angular frequency of small oscillations in the x-z plane.
2. Next consider the sliding conical (also called spherical) pendulum, for which the mass M can move without friction in the horizontal x-y plane. Consider circular motion of both masses for a fixed, not necessarily small, angle θ. What is the angular frequency as a function of θ? Check your result for small θ.
3. Now consider the inverted sliding pendulum (which can only move in x-z plane, i.e. back to the set-up of 1). As it starts falling from rest at $\theta = \pi$, there comes a point where the tension T in the rod becomes zero. Find the value of θ when this happens, assuming for simplicity that M is infinitely large.
4. Same set-up as 3. What is the tension at $\theta = \dfrac{\pi}{2}$ and $\theta = \pi$ when the inverted sliding pendulum is falling from rest at $\theta = \pi$ but this time M and m are arbitrary?

Hint: For an ellipse $x^2/a^2 + y^2/b^2 = 1$, the curvature radius at $(a, 0)$ is b^2/a and at $(0, b)$ is a^2/b.

解答

1. 取水平位移 X 和摆角 θ . m 的坐标为 $x_m = X + l\sin\theta,\ z_m = -l\cos\theta$. 动能 $T = \dfrac{1}{2}M\dot{X}^2 + \dfrac{1}{2}m[(\dot{X} + l\dot{\theta}\cos\theta)^2 + (l\dot{\theta}\sin\theta)^2] = \dfrac{1}{2}(M+m)\dot{X}^2 + \dfrac{1}{2}ml^2\dot{\theta}^2 + ml\dot{X}\dot{\theta}\cos\theta$，势能 $V = -mgl\cos\theta$. 小角度近似 $\cos\theta \approx 1 - \theta^2/2$，保留二阶得 $T \approx \dfrac{1}{2}(M+m)\dot{X}^2 + \dfrac{1}{2}ml^2\dot{\theta}^2 + ml\dot{X}\dot{\theta},\ V \approx -mgl(1-\theta^2/2)$. 拉格朗日方程给出 $(M+m)\dot{X} + ml\dot{\theta} = \text{const.}$ 初始静止，故 $\dot{X} = -\dfrac{ml}{M+m}\dot{\theta}$；代入 θ 方程得 $\dfrac{Mml^2}{M+m}\ddot{\theta} = -mgl\theta$，

即 $\ddot{\theta}+\dfrac{M+m}{M}\dfrac{g}{l}\theta=0$，角频率 $\omega=\sqrt{\dfrac{M+m}{M}\dfrac{g}{l}}$.

2. 系统绕竖直轴以角速度 Ω 转动，m 与竖直方向夹角 θ 固定. 设 M 到转轴距离为 R，质心水平静止得 $MR+m(R+l\sin\theta)=0$ 即 $R=-\dfrac{ml\sin\theta}{M+m}$，$m$ 的圆周半径 $r_m=R+l\sin\theta=\dfrac{Ml\sin\theta}{M+m}$. 竖直方向 $T\cos\theta=mg$，水平方向 $T\sin\theta=m\Omega^2 r_m$. 消去 T 得 $\tan\theta=\dfrac{\Omega^2}{g}\cdot\dfrac{Ml\sin\theta}{M+m}$，故 $\Omega^2=\dfrac{M+m}{M}\dfrac{g}{l}\cos\theta$，即 $\omega(\theta)=\sqrt{\dfrac{M+m}{M}\dfrac{g}{l}\cos\theta}$. 小 θ 时 $\cos\theta\approx1$ 回到第 1 问结果.

3. 能量守恒给出

$$\dot{\theta}^2=\frac{2g}{l}(1+\cos\theta)$$

张力 $T=m\big(g\cos\theta+l\dot{\theta}^2\big)$. 令 $T=0$，解得

$$\cos\theta_0=-\frac{2}{3},\quad \theta_0=\arccos(-2/3)$$

当 $\theta>\theta_0$ 时杆张力变为 0.

4. 由动量守恒

$$\dot{X}=-\frac{ml\cos\theta}{M+m}\,\dot{\phi}$$

将其代入张力表达式

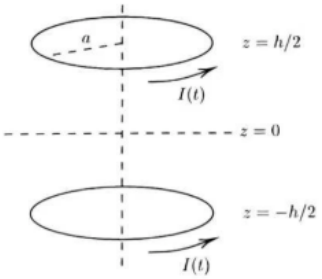
$$T=m\Big(g\cos\theta+l\dot{\theta}^2-\sin\theta\,\dot{X}\,\dot{\theta}\Big)$$

取初始条件 $\theta(0)=\pi$, $\dot{\theta}(0)=0$，计算可得，在 $\theta=\pi$ 时 $T=mg$，在 $\theta=\pi/2$ 时

$$T\Big(\frac{\pi}{2}\Big)=\frac{2\,M\,m\,g}{M+m}$$

2 Problem 2

Two identical circular coils of wire of radius a are separated by a length h. The coils each carry a slowly varying sinusoidal current $I(t)=I_0\cos(\omega t)$. The axis of the coils is aligned with the z-axis and the geometry is centered at $z=0$. (See below)



1. At lowest order in the frequency, a magnetostatic approximation is valid. Using this approximation, show that close to the axis, and near $z=0$, the Taylor series for the axial and radial components of the slightly off-axis magnetic field take the approximate form:

$$B_z\approx B_0+\frac{1}{2}\beta_2\left(z^2-\frac{\rho^2}{2}\right)+\ldots$$

$$B_\rho\approx-\frac{1}{2}\beta_2 z\rho+\ldots$$

where B_0 and β_2 are determined by the Taylor series of the magnetic field on the z-axis:

$$B_z(z) \approx B_0 + \frac{1}{2}\beta_2 z^2 + \dots$$

Here $\rho = \sqrt{x^2 + y^2}$.

2. Using the magnetostatic approximation, determine the magnetic field in the z direction close to the axis of the solenoid, and near $z = 0$, to quadratic order in z and ρ. Describe the magnetic field when $h = a$.
3. Determine the electric field close to the axis of the solenoid at $z = 0$ to the lowest non-trivial order in the frequency and ρ.
4. Briefly answer the following:

   (a) For parts 2 and 3, give an estimate for the size of finite-frequency corrections.

   (b) For part 2, estimate at what large z the magnetostatic approximation breaks down when computing the magnetic field on the z-axis.

解答

1. 在无电流的区域内 $\nabla \times \mathbf{B} = 0$ 且 $\nabla \cdot \mathbf{B} = 0$. 可引入磁标势 Φ 使

$$\mathbf{B} = -\nabla\Phi, \qquad \nabla^2\Phi = 0$$

柱坐标 (ρ, ϕ, z) 中

$$\Phi(z, \rho) = \phi_0(z) + \phi_2(z)\rho^2 + \phi_4(z)\rho^4 + \cdots$$

把它代入拉普拉斯方程并由同阶 ρ 的系数相等得到

$$\phi_2 = -\frac{1}{4}\phi_0'', \qquad \phi_4 = -\frac{1}{16}\phi_2'', \dots$$

于是

$$B_z = -\partial_z\Phi = -\phi_0' - \phi_2'\rho^2 - \cdots = B_0(z) - \frac{B_0''(z)}{4}\,\rho^2 + \cdots$$

$$B_\rho = -\partial_\rho\Phi = -2\phi_2\,\rho - \cdots = -\frac{B_0'(z)}{2}\,\rho + \cdots$$

其中 $B_0(z) \equiv B_z(z, 0) \equiv -\phi_0'(z)$

对 $z = 0$ 的泰勒展开

$$B_0(z) = B_0 + \frac{1}{2}\beta_2\, z^2 + \mathcal{O}(z^4), \qquad B_0''(0) = \beta_2.$$

把上式代回即得到

$$B_z(\rho, z) \approx B_0 + \frac{\beta_2}{2}\left(z^2 - \tfrac{1}{2}\rho^2\right)$$

$$B_\rho(\rho, z) \approx -\frac{\beta_2}{2}\, z\rho$$

这正是题目中给出的近似形式.

2. 单条半径为 a 的闭合线圈在轴上距其中心的距离 z 处的磁场

$$B_l(z) = \frac{\mu_0 I\, a^2}{2\left(a^2 + z^2\right)^{3/2}}$$

两条线圈分别位于 $z = \pm h/2$，于是

$$B_z(z) = \frac{\mu_0 I\, a^2}{2}\left[\frac{1}{\left(a^2 + (z - h/2)^2\right)^{3/2}} + \frac{1}{\left(a^2 + (z + h/2)^2\right)^{3/2}}\right]$$

在 $z=0$ 处

$$B_0=\frac{\mu_0 I\,a^2}{\left(a^2+(h/2)^2\right)^{3/2}}$$

取二阶导数得到

$$\beta_2=-\frac{3\mu_0 I\,a^2\,(a^2-h^2)}{\left(a^2+(h/2)^2\right)^{7/2}}$$

给出在 $z=0$ 附近、ρ 乘到二次项的磁场

$$B_z(\rho,z)\approx B_0+\frac{\beta_2}{2}\left(z^2-\tfrac{1}{2}\rho^2\right),\qquad B_\rho(\rho,z)\approx-\frac{\beta_2}{2}\,z\rho$$

当 $h=a$ 时

$$B_0=\frac{\mu_0 I\,a^2}{\left(a^2+(a/2)^2\right)^{3/2}}=\frac{8\,\mu_0 I}{5\sqrt{5}\,a}$$

而

$$\beta_2=-\frac{3\mu_0 I\,a^2\,(a^2-h^2)}{\left(a^2+(h/2)^2\right)^{7/2}}=0$$

于是

$$B_z(\rho,z)\approx B_0+\mathcal{O}(\rho^4,\,z^4)),\qquad B_\rho(\rho,z)\approx 0$$

结论：线圈间距等于半径时，$z=0$, $\rho\ll a$ 的局域磁场几乎完全均匀，没有二阶磁梯度，可以把它近似为理想等压场.

3. 由法拉第电磁感应律

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

在轴对称、静磁场近似下 $B_z(\rho,z,t)=B_z(\rho,z)\cos\omega t$. 有

$$(\nabla\times\mathbf{E})_z=\frac{1}{\rho}\frac{\partial}{\partial\rho}\big(\rho E_\phi\big)=-\frac{\partial B_z}{\partial t}=\omega B_z(\rho,z)\sin\omega t$$

令 $z=0$，并把 $B_z(\rho,0)=B_0-\frac{\beta_2}{4}\rho^2$ 代入

$$\frac{1}{\rho}\frac{d}{d\rho}\big(\rho E_\phi\big)=\omega\Big(B_0-\frac{\beta_2}{4}\rho^2\Big)\sin\omega t$$

积分得到

$$E_\phi(\rho,0,t)=\frac{\omega B_0}{2}\,\rho\,\sin\omega t-\frac{\omega\beta_2}{16}\,\rho^3\,\sin\omega t+\mathcal{O}(\rho^5).$$

最低非平凡阶

$$\mathbf{E}(\rho,0,t)\approx\frac{\omega B_0}{2}\,\rho\,\sin\omega t\,\hat{\phi}$$

在 $\rho\to 0$ 时，E_ρ, E_z 在此阶都为零. 这正是由变磁通在非常小半径处产生的环形电场.

4. (a) 在低频情形下，磁场与电场的修正可用求解 Helmholtz 方程

$$(\nabla^2+k^2)\mathbf{B}=-\mu_0\,\mathbf{J},\qquad k=\frac{\omega}{c}$$

得到. 对磁场的相对修正

$$\frac{\delta B}{B}\;\sim\;(kR)^2\;=\;\left(\tfrac{\omega R}{c}\right)^2$$

对电场的相对修正

$$\frac{\delta E}{E}\;\sim\;(kR)^2$$

(b) 静磁场近似使用瞬时电流，无时延效应. 当观察点距离至少与波长的量级相当时（相位差 $\approx 1\text{rad}$ ），重叠的波动效应不能忽略. 波长 $\lambda = 2\pi c/\omega$. 失效临界位置 $z_{\text{break}} \sim \dfrac{c}{\omega} \approx \dfrac{\lambda}{2\pi}$.

3 Problem 3

Consider a one-dimensional non-relativistic particle of mass m and kinetic energy E scattering off the potential $U(x)$ composed of two static delta-functions

$$U(x) = \beta[\delta(x) + \delta(x-a)]$$

1. Can the particle tunnel through this barrier without reflection? Explain your answer.
2. If so, at what value(s) of the kinetic energy does this happen?
3. Now consider a three-dimensional non-relativistic particle of mass m and kinetic energy E scattering off the potential $U(\vec{r})$ composed of two static delta-functions

$$U(\vec{r}) = \beta_3(\delta^{(3)}(\vec{r}-\vec{r}_1) + \delta^{(3)}(\vec{r}-\vec{r}_2)). \tag{5}$$

   Suppose that for each of the delta-functions in the above equation the S-wave scattering length $a < 0$.
4. Write explicitly the S-wave bound wave-function near each of the centers. Check that each does not support a bound state for $a < 0$.
5. Can the potential $U(\vec{r})$ with two centers support a bound state? If so, under what conditions? Explain your answer.

Hint: For a hard core potential of radius R, the S-wave scattering length a is defined as

$$\left.\frac{d\log\chi(R)}{dR}\right|_R = -1/a,$$

with $\chi(r)$ the S-wave reduced wave-function $\chi(r) \equiv r\psi(r)$. If $\psi(r)$ corresponds to a bound state, think about the constraints on the scattering length a.

解答

1. 可以. 当入射波能量满足一定条件时，两个 δ 势之间的干涉效应可使反射波完全抵消，实现完全透射. 这类似于光学中的增透膜或量子力学中的共振透射.
2. 利用转移矩阵方法. 单个 δ 势的转移矩阵为

$$M_0 = \begin{pmatrix} 1-i\eta & -i\eta \\ i\eta & 1+i\eta \end{pmatrix}, \quad \eta = \frac{m\beta}{\hbar^2 k}$$

   自由传播矩阵 $P(a) = \text{diag}(e^{ika}, e^{-ika})$. 总转移矩阵 $M_{\text{total}} = M_0 P(a) M_0$.

   入射波 $\begin{pmatrix} 1 \\ r \end{pmatrix}$ 与透射波 $\begin{pmatrix} t \\ 0 \end{pmatrix}$ 满足

$$\begin{pmatrix} t \\ 0 \end{pmatrix} = M_{\text{total}} \begin{pmatrix} 1 \\ r \end{pmatrix}$$

   无反射 ($r = 0$) 要求 $(M_{\text{total}})_{21} = 0$，即

$$i\eta\left[(1-i\eta)e^{ika} + (1+i\eta)e^{-ika}\right] = 0$$

   整理得

$$(1-i\eta)e^{ika} + (1+i\eta)e^{-ika} = 0 \implies e^{-2ika} = -\frac{1-i\eta}{1+i\eta}$$

右边模为 1，可写为 $-e^{-2i\arctan\eta}$，故

$$-2ka=\pi-2\arctan\eta+2n\pi\implies ka=\arctan\eta-\frac{\pi}{2}+n\pi$$

取正切得

$$\tan(ka)=-\cot(\arctan\eta)=-\frac{1}{\eta}=-\frac{\hbar^2k}{m\beta}$$

此即无反射的能量条件，存在一系列分立 k_n 使透射率 $T=1$.

3. 对于单个球对称 δ 势，S 波径向约化波函数 $\chi(r)=r\psi(r)$ 在 $r>0$ 满足自由方程

$$-\frac{\hbar^2}{2m}\chi''(r)=E\chi(r)$$

考虑束缚态 $E=-\dfrac{\hbar^2\kappa^2}{2m}<0$，则 $\chi''(r)=\kappa^2\chi(r)$，有界解为 $\chi(r)=Ce^{-\kappa r}$. 在 $r\to0$ 处，边界条件由散射长度 a 定义为

$$\left.\frac{d\ln\chi(r)}{dr}\right|_{r\to0}=-\frac{1}{a}$$

代入 $\chi(r)=Ce^{-\kappa r}$ 得

$$\frac{\chi'(0)}{\chi(0)}=-\kappa=-\frac{1}{a}\implies\kappa=\frac{1}{a}$$

若 $a<0$，则 $\kappa<0$，此时 $\chi(r)=Ce^{|\kappa|r}$ 在 $r\to0$ 时趋于常数，但在 $r>0$ 时指数增长，不满足平方可积. 因此单个中心在 $a<0$ 时不能支持束缚态.

4. 可以. 两个中心相距 $R=|\vec r_1-\vec r_2|$，总波函数用格林函数叠加

$$\psi(\vec r)=C_1\frac{e^{-\kappa|\vec r-\vec r_1|}}{|\vec r-\vec r_1|}+C_2\frac{e^{-\kappa|\vec r-\vec r_2|}}{|\vec r-\vec r_2|}$$

其中 $\kappa>0$ 待定，束缚态能量 $E=-\hbar^2\kappa^2/(2m)$. 在每个中心附近匹配边界条件.

以 $\vec r_1$ 为中心，令 $\rho=|\vec r-\vec r_1|\to0$，则 $|\vec r-\vec r_2|\to R$. 约化波函数

$$\chi_1(\rho)=\rho\psi\approx C_1(1-\kappa\rho)+C_2\rho\frac{e^{-\kappa R}}{R}$$

其对数导数在 $\rho\to0$ 处

$$\frac{\chi_1'(0)}{\chi_1(0)}=\frac{-C_1\kappa+C_2e^{-\kappa R}/R}{C_1}=-\kappa+\frac{C_2}{C_1}\frac{e^{-\kappa R}}{R}$$

边界条件要求该值等于 $-1/a$，故

$$-\kappa+\frac{C_2}{C_1}\frac{e^{-\kappa R}}{R}=-\frac{1}{a}\implies\frac{C_2}{C_1}\frac{e^{-\kappa R}}{R}=\kappa-\frac{1}{a}$$

同理，在 $\vec r_2$ 处得

$$\frac{C_1}{C_2}\frac{e^{-\kappa R}}{R}=\kappa-\frac{1}{a}$$

两式相乘得

$$\left(\frac{e^{-\kappa R}}{R}\right)^2=\left(\kappa-\frac{1}{a}\right)^2\implies\kappa-\frac{1}{a}=\pm\frac{e^{-\kappa R}}{R}$$

由于 $a<0$，$1/a<0$；$\kappa>0$，左边 $\kappa-1/a>0$. 右边若取负号，则 $\kappa-1/a=-e^{-\kappa R}/R<0$，矛盾. 故只能取正号

$$\kappa-\frac{1}{a}=\frac{e^{-\kappa R}}{R}\implies\frac{1}{a}=\kappa-\frac{e^{-\kappa R}}{R}$$

令 $f(\kappa)=\kappa-\dfrac{e^{-\kappa R}}{R}$，则 $f(0)=-\dfrac{1}{R}<0$，$f(+\infty)=+\infty$，且 $f'(\kappa)=1+e^{-\kappa R}>0$，故 f 严格单调递增. 方程 $f(\kappa)=1/a$ 有正根 κ 当且仅当

$$\frac{1}{a}>f(0)=-\frac{1}{R}\implies-\frac{1}{a}<R$$

由于 $a<0$，记 $|a|=-a$，条件化为

$$|a|>R$$

即两中心距离小于散射长度的绝对值时，存在一个束缚态. 反之，若 $R\geq|a|$，则无束缚态. 这一结果类似于两个吸引中心在足够近时形成分子轨道态，即使每个孤立中心无束缚态.

4 Problem 4

1. Write down the condition for thermodynamic equilibrium between the liquid and gas phases along a liquid-gas coexistence curve.
2. Using your solution to part 1 and taking into account the entropy change at a liquid-gas phase transition, derive the relation for the vapor pressure along a liquid-gas coexistence curve.

解答

1. 沿液 - 气共存曲线，热力学平衡要求温度 T、压强 p 相等且化学势相等，即 $\mu_{\text{liquid}}(T,p)=\mu_{\text{gas}}(T,p)$.
2. 化学势全微分为 $d\mu=-s\,dT+v\,dp$，s 为摩尔熵，v 为摩尔体积. 沿共存曲线 $d\mu_l=d\mu_g$，故 $-s_l dT+v_l dp=-s_g dT+v_g dp$，整理得 $\dfrac{dp}{dT}=\dfrac{s_g-s_l}{v_g-v_l}$. 相变潜热 $L=T(s_g-s_l)$，代入得克拉佩龙方程 $\dfrac{dp}{dT}=\dfrac{L}{T(v_g-v_l)}$. 若气体视为理想气体且 $v_g\gg v_l$，则 $v_g\approx RT/p$，有 $\dfrac{dp}{dT}=\dfrac{Lp}{RT^2}$. 当 L 近似常数时积分得 $p=p_0\exp(-L/(RT))$，即蒸气压力随温度指数上升.

5 Problem 5

Consider the ϕ^4 model with a real scalar field $\phi(x)$ in $3+1$ dimensional Minkowski spacetime with metric $(-,+,+,+)$. Its Lagrangian density is

$$L=-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}m^2\phi^2-\frac{1}{4!}g\phi^4.$$

1. Write down the propagator and the interaction vertex for this model in momentum space.
2. Compute the one-loop correction to the 4-point function using dimensional regularization.
3. Determine the counterterm needed to cancel the divergence and write the renormalized coupling at 1-loop order (you may use minimal subtraction).
4. Write down the renormalization group flow equation for the coupling constant.

Useful formulas:

$$(AB)^{-1}=\int_0^1 dx[xA+(1-x)B]^{-2}$$

$$\int d^dk\frac{1}{(-k^2-2p\cdot k-M^2+i\epsilon)^s}=(-1)^s i\pi^{d/2}\frac{\Gamma(s-d/2)}{\Gamma(s)}(-p^2+M^2-i\epsilon)^{d/2-s}$$

$$\Gamma(z)=1/z-\gamma+O(z)$$

解答

1. 动量空间中实标量场的费曼传播子为 $\Delta(p)=i/(p^2+m^2-i\epsilon)$，其中 $p^2=-p_0^2+\mathbf{p}^2$. 相互作用项 $-\dfrac{g}{4!}\phi^4$ 给出的顶点因子为 $-ig$.

2. 考虑 s 通道，外动量 p_1, p_2 合成 $P = p_1 + p_2$，$s = P^2$，对称因子 $1/2$，振幅为

$$i\mathcal{M}_s = \frac{g^2}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+m^2)((P-k)^2+m^2)} \equiv \frac{g^2}{2}I(s)$$

用费曼参数化 $\dfrac{1}{AB} = \displaystyle\int_0^1 dx[xA+(1-x)B]^{-2}$，令 $A = k^2+m^2$，$B = (P-k)^2+m^2$，配方得分母 $(k-xP)^2+\Delta$，其中 $\Delta = m^2 + x(1-x)s$. 作平移 $l = k - xP$ 得

$$I(s) = \int_0^1 dx \int \frac{d^dl}{(2\pi)^d}\frac{1}{(l^2+\Delta)^2}$$

取 $d = 4-2\epsilon$，得

$$\int \frac{d^dl}{(2\pi)^d}\frac{1}{(l^2+\Delta)^2} = \frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Gamma(2)}\Delta^{d/2-2} = \frac{i}{16\pi^2}\left(\frac{1}{\epsilon} - \gamma + \ln(4\pi) - \ln\Delta + O(\epsilon)\right)$$

故

$$I(s) = \frac{i}{16\pi^2}\int_0^1 dx\left(\frac{1}{\epsilon} - \gamma + \ln(4\pi) - \ln(m^2 + x(1-x)s - i\epsilon)\right)$$

一圈总修正为 $\dfrac{g^2}{2}(I(s)+I(t)+I(u))$，发散部分为 $\dfrac{g^2}{2}\cdot 3\cdot\dfrac{i}{16\pi^2\epsilon} = \dfrac{3ig^2}{32\pi^2\epsilon}$.

3. 在最小减除 (MS) 方案中，令裸耦合 $g_0 = g + \delta g$，要求抵消发散，得 $\delta g = \dfrac{3g^2}{32\pi^2\epsilon}$. 重整化后的耦合常数记为 $g(\mu)$，在 MS 方案下有限.
4. β 函数定义为 $\beta(g) = \mu\dfrac{dg}{d\mu}$. 由 g_0 与 μ 无关可导出 $\beta(g) = \dfrac{3g^2}{16\pi^2}$.

6 Problem 6

The Klein disk model is an analytical description of non-Euclidean geometry in terms of coordinates (x^1, x^2) for points inside a unit disk. The distance $d(\vec{x}, \vec{y})$ between $\vec{x} = (x^1, x^2)$ and $\vec{y} = (y^1, y^2)$ (both inside the disk) is defined by

$$\cosh d(\vec{x}, \vec{y}) = \frac{1 - \vec{x}\cdot\vec{y}}{\sqrt{1-\vec{x}\cdot\vec{x}}\,\sqrt{1-\vec{y}\cdot\vec{y}}},$$

where $\vec{x}\cdot\vec{y} = x^1y^1 + x^2y^2$.

1. Compute the metric components g_{11}, g_{12}, g_{22} from the formula for d by considering $\vec{y}$ near $\vec{x}$ ($y^j = x^j + dx^j$).
2. Introduce standard polar coordinates (r, θ) and write the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ in polar coordinates.
3. Determine the nonvanishing Christoffel symbols in polar coordinates.
4. Compute the component $R_{r\theta r\theta}$ of the Riemann tensor.
5. Show that for the Klein disk model, the Riemann curvature has the form $R_{\mu\nu\rho\sigma} = K(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$ and determine K.
6. Compute the scalar curvature.

解答

1. 令 $\vec{y} = \vec{x} + d\vec{x}$, 记 $r^2 = |\vec{x}|^2$. 展开 $\cosh d \approx 1 + d^2/2$ 并比较 $d\vec{x}$ 的二阶项, 得线元 $ds^2 = \dfrac{d\vec{x}^2}{1-r^2} + \dfrac{(\vec{x}\cdot d\vec{x})^2}{(1-r^2)^2}$. 由此读出 $g_{11} = \dfrac{1}{1-r^2} + \dfrac{(x^1)^2}{(1-r^2)^2}$, $g_{22} = \dfrac{1}{1-r^2} + \dfrac{(x^2)^2}{(1-r^2)^2}$, $g_{12} = g_{21} = \dfrac{x^1x^2}{(1-r^2)^2}$.
2. 设 $x^1 = r\cos\theta$, $x^2 = r\sin\theta$, 则 $d\vec{x}^2 = dr^2 + r^2d\theta^2$, $\vec{x}\cdot d\vec{x} = rdr$. 代入得 $ds^2 = \dfrac{dr^2}{1-r^2} + \dfrac{r^2d\theta^2}{1-r^2} + \dfrac{r^2dr^2}{(1-r^2)^2} = \dfrac{dr^2}{(1-r^2)^2} + \dfrac{r^2}{1-r^2}d\theta^2$. 故 $g_{rr} = 1/(1-r^2)^2$, $g_{\theta\theta} = r^2/(1-r^2)$, $g_{r\theta} = 0$.

3. 计算得 $\Gamma^r_{rr} = \dfrac{2r}{1-r^2}$, $\Gamma^r_{\theta\theta} = -r$, $\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \dfrac{1}{r(1-r^2)}$, 其余为零.
4. 黎曼张量分量 $R_{r\theta r\theta}$ $R_{r\theta r\theta} = g_{rr}R^r{}_{\theta r\theta}$. 计算 $R^r{}_{\theta r\theta} = \partial_r\Gamma^r_{\theta\theta} - \partial_\theta\Gamma^r_{\theta r} + \Gamma^r_{r\lambda}\Gamma^\lambda_{\theta\theta} - \Gamma^r_{\theta\lambda}\Gamma^\lambda_{r\theta}$. 代入已知符号得 $R^r{}_{\theta r\theta} = -\dfrac{r^2}{1-r^2}$, 故 $R_{r\theta r\theta} = \dfrac{1}{(1-r^2)^2}\cdot\left(-\dfrac{r^2}{1-r^2}\right) = -\dfrac{r^2}{(1-r^2)^3}$.
5. 二维流形中黎曼张量可写为 $R_{\mu\nu\rho\sigma} = K(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$. 取分量 $R_{r\theta r\theta} = K(g_{rr}g_{\theta\theta} - g^2_{r\theta}) = Kg_{rr}g_{\theta\theta}$. 代入 $g_{rr}g_{\theta\theta} = \dfrac{1}{(1-r^2)^2}\cdot\dfrac{r^2}{1-r^2} = \dfrac{r^2}{(1-r^2)^3}$, 与第 4 问结果比较得 $K = -1$.
6. 由 $R_{\mu\nu} = g^{\rho\sigma}R_{\mu\rho\nu\sigma} = Kg_{\mu\nu}(2-1) = Kg_{\mu\nu}$, 再缩并得 $R = g^{\mu\nu}R_{\mu\nu} = 2K = -2$. 故标量曲率为 -2 .